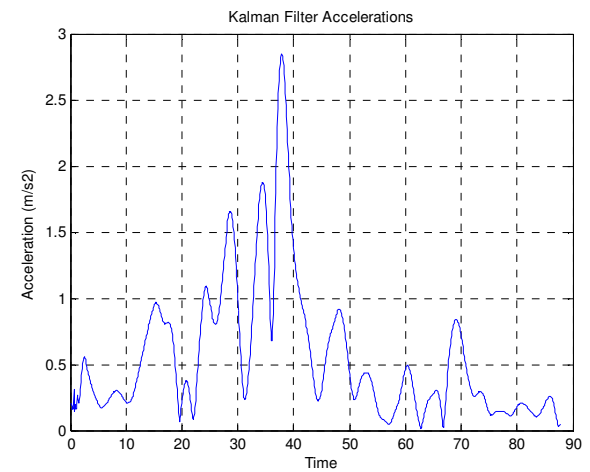
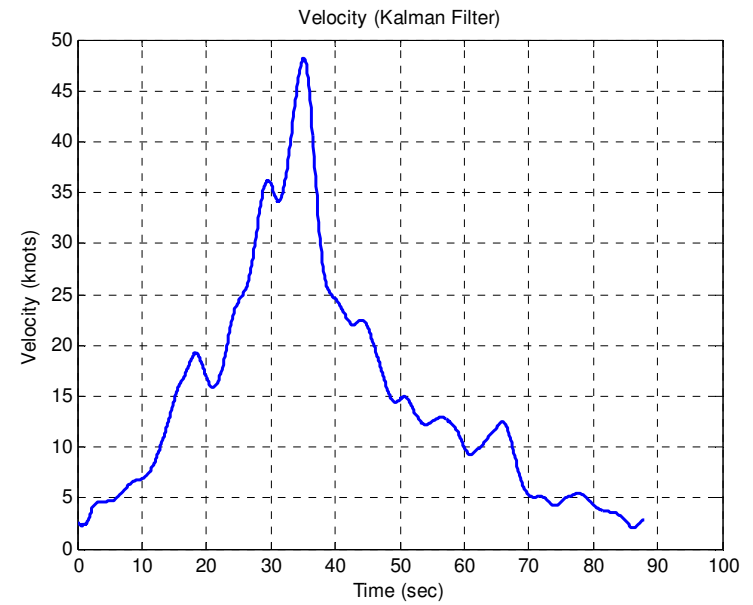


KALMAN FILTERING





steb fisher photography - em: photo@steb.com.au



Kalman Filtering is a least squares estimation process to determine the 'state' of a 'system' from a sequence of measurements at discrete intervals of time. In the previous images the system is the yacht *Macquarie Innovation* (attempting to break the World Sailing Speed Record) and the state is the yacht's position, velocity and acceleration. The measurements were GPS positions (E,N) at 0.1 sec time intervals.

A Kalman Filter is a set of (matrix) equations applied in a recursive sequence.

References for these lectures available at
www.mygeodesy.id.au in the folder Least Squares:

- [1] Least Squares and Kalman Filtering (91 pages)
- [2] Notes on Least Squares (224 pages)
- [3] Combined Least Squares (21 pages)
- [4] Kalman Filter and Surveying Applications (30 pages)

Matrix algebra is used extensively in least squares and Kalman filtering. A useful summary is contained in Appendix A of [2].

REVIEW OF LEAST SQUARES

Least Squares developed by C.F. Gauss in the late 18th century (1795). Gauss concluded:

“... the most probable system of values of the quantities ... will be that in which the sum of the squares of the differences between actually observed and computed values multiplied by numbers that measure the degrees of precision, is a minimum.”

This proposition is often stated as: *the best estimate is that which makes the sum of the squares of the weighted residuals a minimum.*

Here's how this **least squares principle** works.

Consider a set of measurements of a quantity as x_1, x_2, \dots, x_n each having weight w_1, w_2, \dots, w_n . Denote the best estimate of this quantity as q and according to the measurement model:

$$\text{measurement} + \text{residual} = \text{best estimate}$$

we write

$$x_1 + v_1 = q, \quad x_2 + v_2 = q, \quad \dots, \quad x_n + v_n = q$$

which can be re-arranged as

$$v_1 = q - x_1, \quad v_2 = q - x_2, \quad \dots, \quad v_n = q - x_n$$

Now define a function

$$\varphi = \text{sum of squares of weighted residuals} = \sum_{k=1}^n w_k v_k^2$$

and

$$\varphi = \sum_{k=1}^n w_k v_k^2 = w_1 (q - x_1)^2 + w_2 (q - x_2)^2 + \cdots w_n (q - x_n)^2$$

We say that φ is a function of q , or $\varphi = \varphi(q)$ and the minimum value of this function can be found by

equating the derivative $\frac{d\varphi}{dq}$ to zero.

$$\frac{d\varphi}{dq} = 2w_1(q - x_1) + 2w_2(q - x_2) + \cdots 2w_n(q - x_n) = 0$$

Cancelling the 2's, expanding and re-arranging gives the equation for q

$$q = \frac{w_1x_1 + w_2x_2 + \cdots + w_nx_n}{w_1 + w_2 + \cdots + w_n} = \frac{\sum_{k=1}^n w_k x_k}{\sum_{k=1}^n w_k}$$

This is the formula for the weighted mean and q is a best estimate (or a least squares estimate)

Using matrix algebra we write

$$\varphi = \text{sum of squares of weighted residuals} = \mathbf{v}^T \mathbf{W} \mathbf{v}$$

\mathbf{v} is a vector of residuals \mathbf{W} is a symmetric weight matrix. For the weighted mean, \mathbf{W} is an n, n diagonal matrix with weights w_k on the leading diagonal. The superscript τ as in \mathbf{v}^T denotes the transpose of the vector (or matrix).

The least squares principle is:

$$\varphi = \mathbf{v}^T \mathbf{W} \mathbf{v} \rightarrow \min$$

The process of minimizing the function ϕ by equating derivatives to zero is common to all least squares estimations, including Kalman Filtering, yielding equations that can be solved for best estimates.

This example of the weighted mean is taken from [2] Ch. 2

A more general treatment of ‘adjustment problems’ where measurements (or observations) and parameters (or unknowns) are linked by non-linear relationships follows. Detailed explanations are given in [1], [2] and [3].

Suppose the following set of non-linear equations represent the mathematical model in an adjustment

$$F(\hat{\mathbf{l}}, \hat{\mathbf{x}}) = \mathbf{0} \quad (\text{A})$$

\mathbf{l} is a vector of n observations, \mathbf{x} is a vector of u parameters. $\hat{\mathbf{l}}$ and $\hat{\mathbf{v}}$ refer to estimates derived from a least squares process such that

$$\hat{\mathbf{l}} = \mathbf{l} + \mathbf{v} \quad \text{and} \quad \hat{\mathbf{x}} = \mathbf{x} + \delta\mathbf{x} \quad (\text{B})$$

\mathbf{v} and $\delta\mathbf{x}$ are vectors of small corrections (or residuals).

The observations \mathbf{l} have a priori cofactor matrix \mathbf{Q} containing estimates of variances and covariances and the parameters \mathbf{x} are treated as ‘observables’ with a priori cofactor matrix \mathbf{Q}_{xx} . (The idea of treating parameters as observable quantities allows sequential processing).

Cofactor matrices \mathbf{Q} , covariance matrices Σ and weight matrices \mathbf{W} are related as follows

$$\Sigma = \sigma_0^2 \mathbf{Q} \quad \text{and} \quad \mathbf{W} = \mathbf{Q}^{-1}$$

where σ_0^2 is the variance factor

Linearizing (A) using *Taylor's theorem* and ignoring 2nd and higher-order terms gives

$$F(\hat{\mathbf{I}}, \hat{\mathbf{x}}) = F(\mathbf{I}, \mathbf{x}) + \left. \frac{\partial F}{\partial \hat{\mathbf{I}}} \right|_{\mathbf{I}, \mathbf{x}} (\hat{\mathbf{I}} - \mathbf{I}) + \left. \frac{\partial F}{\partial \hat{\mathbf{x}}} \right|_{\mathbf{I}, \mathbf{x}} (\hat{\mathbf{x}} - \delta \mathbf{x}) = \mathbf{0}$$

Using (B) we write this linearized equation as

$$\mathbf{A}_{(m,n)} \mathbf{v}_{(n,1)} + \mathbf{B}_{(m,u)} \delta \mathbf{x}_{(u,1)} = \mathbf{f}_{(m,1)} \quad (\text{C})$$

This equation represents a system of m equations that will be used to estimate u parameters from n observations.

$$n \geq m \geq u$$

$\mathbf{A}_{(m,n)} = \frac{\partial F}{\partial \hat{\mathbf{I}}} \Big|_{l,x}$ and $\mathbf{B}_{(m,u)} = \frac{\partial F}{\partial \hat{\mathbf{x}}} \Big|_{l,x}$ are coefficient matrices of partial derivatives and $\mathbf{f}_{(m,1)} = -\{F(\mathbf{l}, \mathbf{x})\}$ is a vector of numeric terms.

The least squares solution of equations (C) can be obtained by minimizing the function

$$\varphi = \mathbf{v}^T \mathbf{W} \mathbf{v} + \delta \mathbf{x}^T \mathbf{W}_{xx} \delta \mathbf{x} - 2\mathbf{k}^T (\mathbf{A} \mathbf{v} + \mathbf{B} \delta \mathbf{x} - \mathbf{f})$$

\mathbf{k} are a vector of *Lagrange multipliers* (at present unknown) and the 2 is added for convenience.

$\varphi = \varphi(\mathbf{v}, \delta\mathbf{x})$ and the partial derivatives equated to zero are

$$\frac{\partial \varphi}{\partial \mathbf{v}} = 2\mathbf{v}^T \mathbf{W} - 2\mathbf{k}^T \mathbf{A} = \mathbf{0}^T \quad \text{and} \quad \frac{\partial \varphi}{\partial \delta\mathbf{x}} = 2\delta\mathbf{x}^T \mathbf{W}_{xx} - 2\mathbf{k}^T \mathbf{B} = \mathbf{0}^T$$

These equations can be simplified by dividing both sides by 2, transposing and changing signs to give

$$-\mathbf{W}\mathbf{v} + \mathbf{A}^T \mathbf{k} = \mathbf{0} \quad \text{and} \quad \mathbf{W}_{xx} \delta\mathbf{x} + \mathbf{B}^T \mathbf{k} = \mathbf{0}$$

Note that $(\mathbf{ABC})^T = \mathbf{C}^T \mathbf{B}^T \mathbf{A}^T$

Combining these equations with (C) and arranging in a matrix gives

$$\begin{bmatrix} -\mathbf{W} & \mathbf{A}^T & \mathbf{0} \\ \mathbf{A} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{B}^T & -\mathbf{W}_{xx} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{k} \\ \delta \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \\ \mathbf{0} \end{bmatrix} \quad (\text{D})$$

Equation (D) can be solved by the following **matrix reduction process**.

Consider the partitioned matrix equation $\mathbf{P}\mathbf{y} = \mathbf{u}$ given as

$$\left[\begin{array}{c|c} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \hline \mathbf{P}_{21} & \mathbf{P}_{22} \end{array} \right] \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \quad (\text{i})$$

which can be expanded to give

$$\mathbf{P}_{11}\mathbf{y}_1 + \mathbf{P}_{12}\mathbf{y}_2 = \mathbf{u}_1 \text{ or } \mathbf{y}_1 = \mathbf{P}_{11}^{-1}(\mathbf{u}_1 - \mathbf{P}_{12}\mathbf{y}_2) \quad (\text{ii})$$

Eliminating \mathbf{y}_1 by substituting (ii) into (i) gives

$$\left[\begin{array}{c|c} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \hline \mathbf{P}_{21} & \mathbf{P}_{22} \end{array} \right] \left[\begin{array}{c} \mathbf{P}_{11}^{-1}(\mathbf{u}_1 - \mathbf{P}_{12}\mathbf{y}_2) \\ \hline \mathbf{y}_2 \end{array} \right] = \left[\begin{array}{c} \mathbf{u}_1 \\ \hline \mathbf{u}_2 \end{array} \right]$$

Expanding the matrix equation gives

$$\mathbf{P}_{21}\mathbf{P}_{11}^{-1}(\mathbf{u}_1 - \mathbf{P}_{12}\mathbf{y}_2) + \mathbf{P}_{22}\mathbf{y}_2 = \mathbf{u}_2$$

$$\mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{u}_1 - \mathbf{P}_{21}\mathbf{P}_{11}^{-1}\mathbf{P}_{12}\mathbf{y}_2 + \mathbf{P}_{22}\mathbf{y}_2 = \mathbf{u}_2$$

and an expression for \mathbf{y}_2 is given by

$$\left(\mathbf{P}_{22} - \mathbf{P}_{21} \mathbf{P}_{11}^{-1} \mathbf{P}_{12} \right) \mathbf{y}_2 = \mathbf{u}_2 - \mathbf{P}_{21} \mathbf{P}_{11}^{-1} \mathbf{u}_1 \quad (\text{iii})$$

Applying this **matrix reduction process** to (D) – see [1] and [3] – yields the following equations

$$\delta \mathbf{x} = \left(\mathbf{N} + \mathbf{W}_{xx} \right)^{-1} \mathbf{t} \quad (\text{x})$$

$$\mathbf{k} = \mathbf{W}_e \left(\mathbf{f} - \mathbf{B} \delta \mathbf{x} \right) \quad (\text{y})$$

$$\mathbf{v} = \mathbf{Q} \mathbf{A}^T \mathbf{k} \quad (\text{z})$$

$$\hat{\mathbf{x}} = \mathbf{x} + \delta \mathbf{x}, \hat{\mathbf{I}} = \mathbf{I} + \mathbf{v}$$

where

$$\mathbf{Q}_e = \mathbf{A} \mathbf{Q} \mathbf{A}^T, \mathbf{W}_e = \mathbf{Q}_e^{-1}, \mathbf{N} = \mathbf{B}^T \mathbf{W}_e \mathbf{B}, \mathbf{t} = \mathbf{B}^T \mathbf{W}_e \mathbf{f}$$

Equations (x), (y) and (z) are the solutions to **Combined Least Squares** problems. If the problems involve non-linear functions (as they often do in surveying) then the process of solution may be iterative and repeated until corrections $\delta\mathbf{x}$ become negligible. References [1], [2] and [3] have examples of iterative solutions.

An important and useful property of least squares solutions is that estimates of the precisions of parameters $\hat{\mathbf{x}}$, observations $\hat{\mathbf{l}}$ and residuals \mathbf{v} are easily obtained. These estimates are contained in the cofactor matrices

$$\mathbf{Q}_{\hat{x}\hat{x}} = (\mathbf{N} + \mathbf{W}_{xx})^{-1}$$

$$\mathbf{Q}_{\hat{y}\hat{y}} = \mathbf{Q} + \mathbf{Q}\mathbf{A}^T \mathbf{W}_e \mathbf{B} \mathbf{Q}_{\hat{x}\hat{x}} \mathbf{B}^T \mathbf{W}_e \mathbf{A} \mathbf{Q} - \mathbf{Q}\mathbf{A}^T \mathbf{W}_e \mathbf{A} \mathbf{Q}$$

$$\mathbf{Q}_{vv} = \mathbf{Q} - \mathbf{Q}_{\hat{y}\hat{y}}$$

The **matrix reduction process** outlined above (a matrix partitioning solution to a set of equations) is important. It is used in the development of the Kalman Filter equations where a large hyper-matrix is reduced by repeated applications of the reduction process (see reference [1])

The cofactor matrices $\mathbf{Q}_{\hat{x}\hat{x}}$, $\mathbf{Q}_{\hat{y}\hat{y}}$ and \mathbf{Q}_{vv} are obtained using **Propagation of Variances** (or cofactors) which is an important tool in analysis of measurements.

Suppose that computed quantities \mathbf{y} are linear functions of variables \mathbf{x} (with cofactor matrix \mathbf{Q}_{xx}) and constant terms \mathbf{b} (\mathbf{A} is a coefficient matrix)

$$\mathbf{y} = \mathbf{Ax} + \mathbf{b}$$

then, Propagation of Variances gives

$$\mathbf{Q}_{yy} = \mathbf{AQ}_{xx}\mathbf{A}^T \quad (*)$$

If \mathbf{y} are non-linear functions of \mathbf{x} then Propagation of Variances gives

$$\mathbf{Q}_{yy} = \mathbf{J}_{yx} \mathbf{Q}_{xx} \mathbf{J}_{yx}^T \quad (**)$$

where \mathbf{J}_{yx} is the Jacobian matrix of partial derivatives.

For most of the cofactor propagation used in least squares we are dealing with linear or linearized functions so (*) is most often used.

Cofactor propagation is very important in the development of the Kalman Filter equations.

The general case of Combined Least Squares [see solutions (x), (y) and (z) above] may be simplified into the following three cases:

[1] Combined Least Squares

For most least squares problems, the parameters \mathbf{x} are not 'observables' and $\mathbf{W}_{xx} = \mathbf{0}$. The system of equations is

$$\mathbf{A}\mathbf{v} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}$$

and the solutions are

$$\begin{aligned}\delta \mathbf{x} &= \mathbf{N}^{-1} \mathbf{t} \\ \mathbf{k} &= \mathbf{W}_e (\mathbf{f} - \mathbf{B} \delta \mathbf{x}) \\ \mathbf{v} &= \mathbf{Q} \mathbf{A}^T \mathbf{k} \\ \hat{\mathbf{x}} &= \mathbf{x}^0 + \delta \mathbf{x}, \hat{\mathbf{I}} = \mathbf{I} + \mathbf{v}\end{aligned}$$

where

$$\mathbf{Q}_e = \mathbf{A} \mathbf{Q} \mathbf{A}^T, \mathbf{W}_e = \mathbf{Q}_e^{-1}, \mathbf{N} = \mathbf{B}^T \mathbf{W}_e \mathbf{B}, \mathbf{t} = \mathbf{B}^T \mathbf{W}_e \mathbf{f}$$

[See reference [1] Example 2]

[2] Indirect Least Squares (Parametric case)

In this case $\mathbf{W}_{xx} = \mathbf{0}$ and $\mathbf{A} = \mathbf{I}$ (the Identity matrix).

The system of equations is

$$\mathbf{v} + \mathbf{B}\delta\mathbf{x} = \mathbf{f}$$

and the solutions are

$$\delta\mathbf{x} = \mathbf{N}^{-1}\mathbf{t}$$

$$\mathbf{v} = \mathbf{f} - \mathbf{B}\delta\mathbf{x}$$

$$\hat{\mathbf{x}} = \mathbf{x}^0 + \delta\mathbf{x}, \hat{\mathbf{I}} = \mathbf{I} + \mathbf{v}$$

where $\mathbf{N} = \mathbf{B}^T \mathbf{W} \mathbf{B}$, $\mathbf{t} = \mathbf{B}^T \mathbf{W} \mathbf{f}$

[See reference [1] Examples 1, 3 and 4]

[3] **Observations Only Least Squares** (Condition case)

In this case $\mathbf{W}_{xx} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$. The system of equations is

$$\mathbf{A}\mathbf{v} = \mathbf{f}$$

and the solutions are

$$\mathbf{k} = \mathbf{W}_e \mathbf{f}$$

$$\mathbf{v} = \mathbf{Q}\mathbf{A}^T \mathbf{k}$$

$$\hat{\mathbf{l}} = \mathbf{l} + \mathbf{v}$$

[See reference [1], Example 3]

KALMAN FILTER

The Kalman Filter is an estimation process developed by Rudolf E. Kalman in the early 1960's. It's first practical use was in the navigation system of the Apollo space craft. The Kalman Filter can be regarded as a logical extension of Gauss' original least squares technique and the derivation of the equations of the Filter set out below follows the 'usual' surveying development of minimizing a function of sums of squares of weighted residuals.

For this development it is useful to consider a ship moving in a navigation channel. At regular intervals of time (epochs t_k) a measurement system on board determines the distances to known navigation beacons. We wish to determine the position and velocity of the ship as it moves down the channel.

Suppose we write a system of equations as

$$\begin{aligned}
 \mathbf{v}_{k-1} + \mathbf{B}\mathbf{x}_{k-1} &= \mathbf{f}_{k-1} && \text{primary at } t_{k-1} \\
 \mathbf{v}_k + \mathbf{B}\mathbf{x}_k &= \mathbf{f}_k && \text{primary at } t_k \\
 \mathbf{x}_k &= \mathbf{T}\mathbf{x}_{k-1} + \mathbf{v}_m && \text{secondary or dynamic}
 \end{aligned} \tag{a}$$

- x** is the state vector containing the parameters of the system, in this case, the position (E, N) and the velocity (\dot{E}, \dot{N}) , four parameters
- v** is the vector of residuals associated with the measurements \mathbf{l} such that $\hat{\mathbf{l}} = \mathbf{l} + \mathbf{v}$
- B** is a coefficient matrix
- f** is a vector of numeric terms derived from the measurements \mathbf{l}
- T** is the Transition matrix
- v_m** is a vector of residuals associated with the dynamic or secondary model

The measurements \mathbf{l} and the model corrections \mathbf{v}_m have associated cofactor matrices \mathbf{Q} and \mathbf{Q}_m and weight matrices $\mathbf{W} = \mathbf{Q}^{-1}$

It is useful to assume the vector \mathbf{v}_m is the product of two matrices

$$\mathbf{v}_m = \mathbf{H}\mathbf{w} \quad (\text{b})$$

where \mathbf{H} is a coefficient matrix and \mathbf{w} is vector of quantities known as the system driving noise.

Assuming the system driving noise \mathbf{w} has a cofactor matrix \mathbf{Q}_w , applying Propagation of Variances to (b)

gives the cofactor matrix of the system of the secondary model

$$\mathbf{Q}_m = \mathbf{H}\mathbf{Q}_w\mathbf{H}^T$$

Applying the least squares principle to equations (a), the function φ to be minimized is

$$\begin{aligned}\varphi = & \mathbf{v}_{k-1}^T \mathbf{W}_{k-1} \mathbf{v}_{k-1} + \mathbf{v}_k^T \mathbf{W}_k \mathbf{v}_k + \mathbf{v}_m^T \mathbf{W}_m \mathbf{v}_m \\ & - 2\mathbf{k}_1^T \left(\mathbf{v}_{k-1} + \mathbf{B}_{k-1} \mathbf{x}_{k-1} - \mathbf{f}_{k-1} \right) \\ & - 2\mathbf{k}_2^T \left(\mathbf{v}_k + \mathbf{B}_k \mathbf{x}_k - \mathbf{f}_k \right) \\ & - 2\mathbf{k}_3^T \left(\mathbf{x}_k - \mathbf{T} \mathbf{x}_{k-1} - \mathbf{v}_m \right)\end{aligned}$$

Now $\varphi = \varphi(\mathbf{v}_{k-1}, \mathbf{v}_k, \mathbf{v}_m, \mathbf{x}_{k-1}, \mathbf{x}_k)$ and differentiating φ with respect to the variables gives

$$\begin{aligned} \frac{\partial \varphi}{\partial \mathbf{v}_{k-1}} &= 2\mathbf{v}_{k-1}^T \mathbf{W}_{k-1} - 2\mathbf{k}_1^T = \mathbf{0} & \frac{\partial \varphi}{\partial \mathbf{v}_k} &= 2\mathbf{v}_k^T \mathbf{W}_k - 2\mathbf{k}_2^T = \mathbf{0} \\ \frac{\partial \varphi}{\partial \mathbf{v}_m} &= 2\mathbf{v}_m^T \mathbf{W}_m + 2\mathbf{k}_3^T = \mathbf{0} & \frac{\partial \varphi}{\partial \mathbf{x}_{k-1}} &= -2\mathbf{k}_1^T \mathbf{B}_{k-1} + 2\mathbf{k}_3^T \mathbf{T} = \mathbf{0} \\ \frac{\partial \varphi}{\partial \mathbf{x}_k} &= -2\mathbf{k}_2^T \mathbf{B}_k - 2\mathbf{k}_3^T = \mathbf{0} \end{aligned}$$

Transposing, re-arranging and dividing by 2 gives

$$\mathbf{W}_{k-1} \mathbf{v}_{k-1} - \mathbf{k}_1 = \mathbf{0} \quad -\mathbf{B}_{k-1}^T \mathbf{k}_1 + \mathbf{T}^T \mathbf{k}_3 = \mathbf{0}$$

$$\mathbf{W}_k \mathbf{v}_k - \mathbf{k}_2 = \mathbf{0} \quad -\mathbf{B}_{k-1}^T \mathbf{k}_2 - \mathbf{k}_3 = \mathbf{0}$$

$$\mathbf{W}_m \mathbf{v}_m + \mathbf{k}_3 = \mathbf{0}$$

Combining these equations with equations (a) – the primary and secondary model equations – in the form of a hyper-matrix gives

$$\left[\begin{array}{ccc|cccc}
 \mathbf{W}_{k-1} & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & 0 \\
 0 & \mathbf{W}_k & 0 & 0 & -\mathbf{I} & 0 & 0 & 0 \\
 0 & 0 & \mathbf{W}_m & 0 & 0 & \mathbf{I} & 0 & 0 \\
 \hline
 0 & 0 & 0 & -\mathbf{B}_{k-1}^T & 0 & \mathbf{T}^T & 0 & 0 \\
 0 & 0 & 0 & 0 & -\mathbf{B}_k^T & -\mathbf{I} & 0 & 0 \\
 \mathbf{I} & 0 & 0 & 0 & 0 & 0 & \mathbf{B}_{k-1} & 0 \\
 0 & \mathbf{I} & 0 & 0 & 0 & 0 & 0 & \mathbf{B}_k \\
 0 & 0 & -\mathbf{I} & 0 & 0 & 0 & -\mathbf{T} & \mathbf{I}
 \end{array} \right]
 \begin{bmatrix}
 \mathbf{v}_{k-1} \\
 \mathbf{v}_k \\
 \mathbf{v}_m \\
 \mathbf{k}_1 \\
 \mathbf{k}_2 \\
 \mathbf{k}_3 \\
 \mathbf{x}_{k-1} \\
 \mathbf{x}_k
 \end{bmatrix}
 =
 \begin{bmatrix}
 0 \\
 0 \\
 0 \\
 0 \\
 0 \\
 \mathbf{f}_{k-1} \\
 \mathbf{f}_k \\
 0
 \end{bmatrix}$$

hyper-matrix (c)

Partial solution \mathbf{x}'_{k-1}

Using only the observations \mathbf{l}_{k-1} at time t_{k-1} a partial solution \mathbf{x}'_{k-1} can be obtained from the hyper-matrix (c) by deleting all matrices associated with the primary model at t_k and the secondary model. This gives (after some re-arrangement)

$$\begin{bmatrix} -\mathbf{W}_{k-1} & \mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{B}_{k-1} \\ \mathbf{0} & \mathbf{B}_{k-1}^T & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{v}'_{k-1} \\ \mathbf{k}'_1 \\ \mathbf{x}'_{k-1} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f}_{k-1} \\ \mathbf{0} \end{bmatrix}$$

This is identical in form to previous equation (D) and the solution is

$$\mathbf{x}'_{k-1} = \mathbf{N}_{k-1}^{-1} \mathbf{t}_{k-1} \quad (\text{d})$$

where

$$\mathbf{N}_{k-1} = \mathbf{B}_{k-1}^T \mathbf{W}_{k-1} \mathbf{B}_{k-1} \quad \mathbf{t}_{k-1} = \mathbf{B}_{k-1}^T \mathbf{W}_{k-1} \mathbf{f}_{k-1}$$

with the cofactor matrix of the partial solution

$$\mathbf{Q}_{\mathbf{x}'_{k-1}} = \mathbf{N}_{k-1}^{-1}$$

Solution for $\hat{\mathbf{x}}_k$

The hyper-matrix (c) is partitioned in a way that allows for the elimination of the vectors $\mathbf{v}_{k-1}, \mathbf{v}_k, \mathbf{v}_m$ by the reduction process set out previously. And after this the parameters \mathbf{x}_{k-1} are eliminated (by the same process) yielding

$$\left[\begin{array}{cc|cc} \mathbf{Q}_m + \mathbf{T}\mathbf{N}_{k-1}^{-1}\mathbf{T}^T & \mathbf{I} & \mathbf{0} & \\ \hline -\mathbf{I} & \mathbf{0} & -\mathbf{B}_k^T & \\ \mathbf{0} & \mathbf{B}_k & \mathbf{Q}_k & \end{array} \right] \begin{bmatrix} \mathbf{k}_3 \\ \mathbf{x}_k \\ \mathbf{k}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{T}\mathbf{N}_{k-1}^{-1}\mathbf{t}_{k-1} \\ \mathbf{0} \\ \mathbf{f}_k \end{bmatrix} \quad (\text{e})$$

Now, following equations (a) we write the predicted state vector $\mathbf{x}'_k = \mathbf{T}\mathbf{x}'_{k-1}$ and substituting (d) gives

$$\mathbf{x}'_k = \mathbf{T}\mathbf{N}_{k-1}^{-1}\mathbf{t}_{k-1} \quad (\text{f})$$

And applying Propagation of Variances gives (after some reduction) the cofactor matrix of the predicted residuals

$$*** \quad \mathbf{Q}_{x'_k} = \mathbf{T}\mathbf{Q}_{x'_{k-1}}\mathbf{T} + \mathbf{Q}_m \quad (\text{g})$$

The quantities in (f) and (g) appear in (e) and we may write

$$\left[\begin{array}{c|cc} \mathbf{Q}_{x'_k} & \mathbf{I} & \mathbf{0} \\ \hline -\mathbf{I} & \mathbf{0} & -\mathbf{B}_k^T \\ \hline \mathbf{0} & \mathbf{B}_k & \mathbf{Q}_k \end{array} \right] \begin{bmatrix} \mathbf{k}_3 \\ \mathbf{x}_k \\ \mathbf{k}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{x}'_k \\ \mathbf{0} \\ \mathbf{f}_k \end{bmatrix}$$

We may apply the matrix reduction process to this matrix to obtain (after some manipulation)

$$\mathbf{x}_k = \mathbf{x}'_k + \mathbf{Q}_{x'_k} \mathbf{B}_k^T \left(\mathbf{Q}_k + \mathbf{B}_k \mathbf{Q}_{x'_k} \mathbf{B}_k^T \right)^{-1} \left(\mathbf{f}_k - \mathbf{B}_k \mathbf{x}'_k \right)$$

In Kalman Filtering, the expression

$$*** \quad \mathbf{K} = \mathbf{Q}_{x'_k} \mathbf{B}_k^T \left(\mathbf{Q}_k + \mathbf{B}_k \mathbf{Q}_{x'_k} \mathbf{B}_k^T \right)^{-1}$$

is called the gain matrix

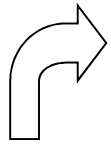
Using this result gives the filtered state vector

$$*** \quad \hat{\mathbf{x}}_k = \mathbf{x}'_k + \mathbf{K}(\mathbf{f}_k - \mathbf{B}_k \mathbf{x}'_k)$$

Applying Propagation of Variances to this equation gives (after some manipulation)

$$\begin{aligned} *** \quad \mathbf{Q}_{\hat{x}_k} &= (\mathbf{I} - \mathbf{K} \mathbf{B}_k) \mathbf{Q}_{x'_k} (\mathbf{I} - \mathbf{K} \mathbf{B}_k)^T + \mathbf{K} \mathbf{Q}_k \mathbf{K}^T \\ &= (\mathbf{I} - \mathbf{K} \mathbf{B}_k)^T \mathbf{Q}_{x'_k} \end{aligned}$$

THE KALMAN FILTER



(1) Compute the predicted state vector at t_k

$$\mathbf{x}'_k = \mathbf{T}\hat{\mathbf{x}}_{k-1}$$

(2) Compute the predicted state cofactor matrix at t_k

$$\mathbf{Q}_{x'_k} = \mathbf{T}\mathbf{Q}_{\hat{x}_{k-1}}\mathbf{T} + \mathbf{Q}_m$$

(3) Compute the Kalman *Gain* matrix

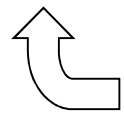
$$\mathbf{K} = \mathbf{Q}_{x'_k}\mathbf{B}_k^T \left(\mathbf{Q}_k + \mathbf{B}_k\mathbf{Q}_{x'_k}\mathbf{B}_k^T \right)^{-1}$$

(4) Compute the filtered state vector by updating the predicted state with the measurements at t_k

$$\mathbf{x}_k = \mathbf{x}'_k + \mathbf{K}(\mathbf{f}_k - \mathbf{B}_k \mathbf{x}'_k)$$

(5) Compute the filtered state cofactor matrix

$$\mathbf{Q}_{\hat{\mathbf{x}}_k} = (\mathbf{I} - \mathbf{K}\mathbf{B}_k) \mathbf{Q}_{\mathbf{x}'_k}$$



Go to step (1) and repeat the process for the next measurement epoch t_{k+1}

MATRIX REDUCTION EXERCISE

If a matrix can be partitioned as follows

$$\left[\begin{array}{c|c} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \hline \mathbf{P}_{21} & \mathbf{P}_{22} \end{array} \right] \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \quad (\text{i})$$

then (after some manipulation) \mathbf{y}_1 is eliminated and

$$\left(\mathbf{P}_{22} - \mathbf{P}_{21} \mathbf{P}_{11}^{-1} \mathbf{P}_{12} \right) \mathbf{y}_2 = \mathbf{u}_2 - \mathbf{P}_{21} \mathbf{P}_{11}^{-1} \mathbf{u}_1 \quad (\text{ii})$$

Use this matrix reduction process to obtain an expression for $\delta \mathbf{x}$ from the system of equations expressed in matrix form as

$$\begin{bmatrix} -\mathbf{W} & \mathbf{A}^T & \mathbf{0} \\ \mathbf{A} & \mathbf{0} & \mathbf{B} \\ \mathbf{0} & \mathbf{B}^T & -\mathbf{W}_{xx} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{k} \\ \delta \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \\ \mathbf{0} \end{bmatrix} \quad (\text{A})$$

SOLUTION

Partitioning (A) in the same way as (i)

$$\left[\begin{array}{c|cc} -\mathbf{W} & \mathbf{A}^T & \mathbf{0} \\ \hline \mathbf{A} & \mathbf{0} & \mathbf{B} \\ \hline \mathbf{0} & \mathbf{B}^T & -\mathbf{W}_{xx} \end{array} \right] \begin{bmatrix} \mathbf{v} \\ \mathbf{k} \\ \delta \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

then eliminating \mathbf{v} by applying (ii) gives

$$\left[\begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{B}^T & -\mathbf{W}_{xx} \end{bmatrix} - \begin{bmatrix} \mathbf{A} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} -\mathbf{W}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{A}^T & \mathbf{0} \end{bmatrix} \right] \begin{bmatrix} \mathbf{k} \\ \delta \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{A} \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} -\mathbf{W}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \end{bmatrix}$$

Remembering that $\mathbf{Q} = \mathbf{W}^{-1}$ the equation can be simplified as

$$\left[\begin{array}{c|c} \mathbf{AQA}^T & \mathbf{B} \\ \hline \mathbf{B}^T & -\mathbf{W}_{xx} \end{array} \right] \begin{bmatrix} \mathbf{k} \\ \delta \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \mathbf{0} \end{bmatrix} \quad (\text{B})$$

Again, applying (ii) to the partitioned equation (B) gives

$$\left(-\mathbf{W}_{xx} - \mathbf{B}^T (\mathbf{AQA}^T)^{-1} \mathbf{B} \right) \delta \mathbf{x} = \mathbf{0} - \mathbf{B}^T (\mathbf{AQA}^T)^{-1} \mathbf{f}$$

and re-arranging gives the normal equations

$$\left(\mathbf{B}^T (\mathbf{AQA}^T)^{-1} \mathbf{B} + \mathbf{W}_{xx} \right) \delta \mathbf{x} = \mathbf{B}^T (\mathbf{AQA}^T)^{-1} \mathbf{f}$$

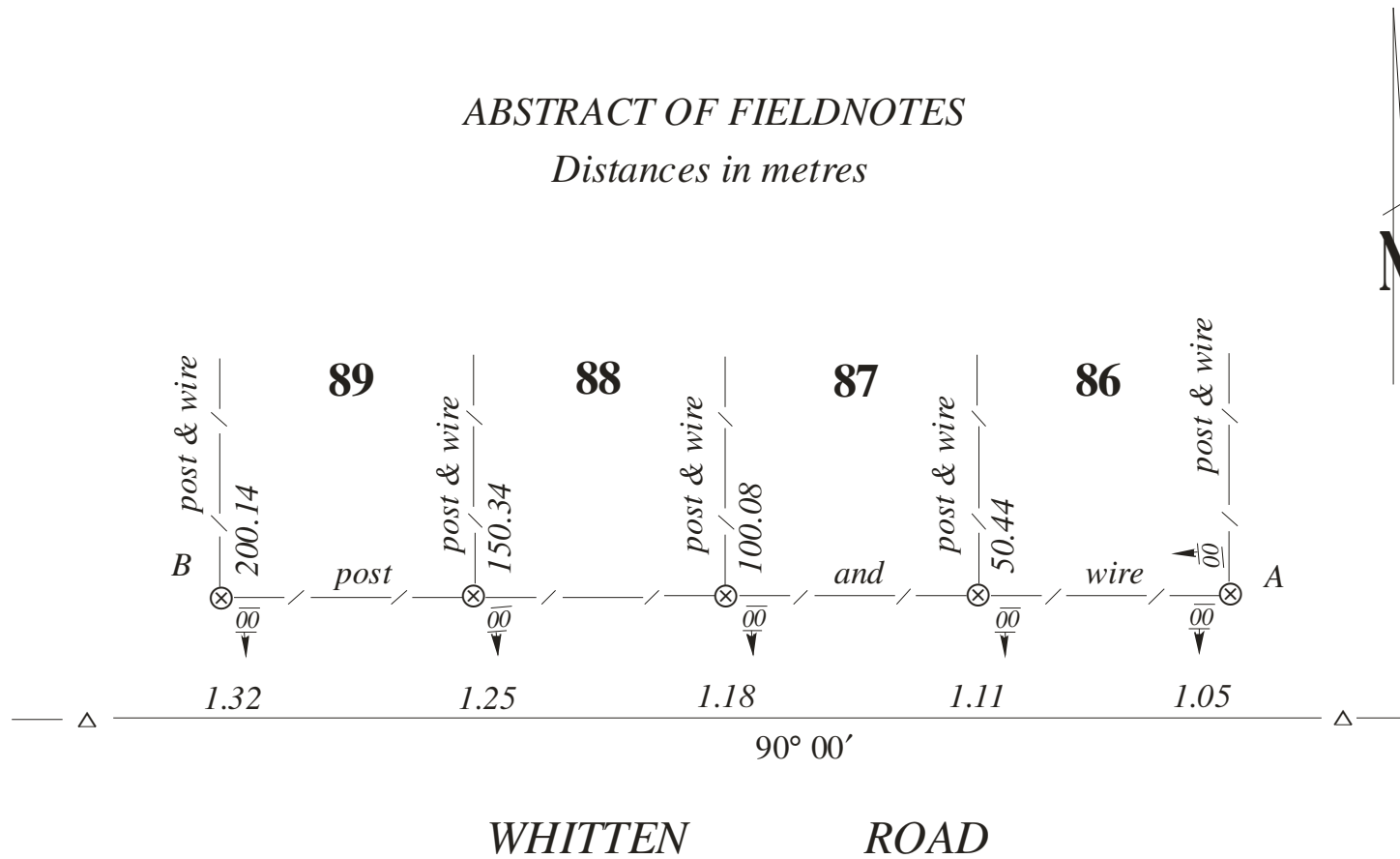
LINE OF BEST FIT EXERCISE

The diagram overleaf shows part of an Abstract of Field Notes with offsets from occupation to a traverse line in Whitten Street. The bearing of the traverse line is $90^{\circ} 00'$

Use **Indirect Least Squares** (and *least_squares.m*) to determine the bearing of the line of best fit through the occupation in Whitten Street. You may consider the chainages (linear distances along Whitten Street) to be free of error.

ABSTRACT OF FIELDNOTES

Distances in metres



SOLUTION

The equation of the Line of Best Fit is: $y = bx + c$.

The y -values are the offsets, the x -values are derived from the chainages and c is a constant.

Reverse the chainages to create distances from B .

These will be the x -values (considered error free).

The offsets (the y -values) are the measurements, considered to be affected by small random errors, and assumed to be of equal precision.

A Table of values is shown overleaf.

Point	x (m)	y (m)	weight w
1 (B)	0.000	1.320	1
2	49.800	1.250	1
3	100.060	1.180	1
4	149.700	1.110	1
5 (A)	200.140	1.050	1

The observation equation (or measurement model) is

$$y_k + v_k = bx_k + c$$

The set of equations for each measurement are

$$y_1 + v_1 = bx_1 + c$$

$$y_2 + v_2 = bx_2 + c$$

$$y_3 + v_3 = bx_3 + c$$

$$y_4 + v_4 = bx_4 + c$$

$$y_5 + v_5 = bx_5 + c$$

This equations can be re-arranged in the matrix form form $\mathbf{v} + \mathbf{B}\mathbf{x} = \mathbf{f}$ where the residuals \mathbf{v} and the ‘unknowns’ (b, c) in \mathbf{x} and the coefficient matrix \mathbf{B} are on the left-hand-side of the equals sign.

The measurements (or functions of measurements) are on the right-hand-side forming the vector of numeric terms **f**.

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} + \begin{bmatrix} -x_1 & -1 \\ -x_2 & -1 \\ -x_3 & -1 \\ -x_4 & -1 \\ -x_5 & -1 \end{bmatrix} \begin{bmatrix} b \\ c \end{bmatrix} = \begin{bmatrix} -y_1 \\ -y_2 \\ -y_3 \\ -y_4 \\ -y_5 \end{bmatrix}$$

Now create an ASCII text file that can be used in MATLAB function *least_squares.m*

This text file has a standard format

Save the text file as *best_fit_exercise.txt*

```
% Data file for function "least_squares.m"
% Line of Best Fit Exercise
%  B(1)    B(2)    f      w
      0.0      -1    -1.320    1
    -49.800    -1    -1.250    1
   -100.060    -1    -1.180    1
   -149.700    -1    -1.110    1
   -200.140    -1    -1.050    1
```

MATLAB function *least_squares.m* uses the 'standard solution' for Indirect Least Squares

$$\mathbf{x} = \mathbf{N}^{-1}\mathbf{t}$$

$$\mathbf{v} = \mathbf{f} - \mathbf{B}\mathbf{x}$$

$$\hat{\mathbf{l}} = \mathbf{l} + \mathbf{v}$$

where $\mathbf{N} = \mathbf{B}^T \mathbf{W} \mathbf{B}$, $\mathbf{t} = \mathbf{B}^T \mathbf{W} \mathbf{f}$

The function will read the data text file and place the results in another text file with the extension .out

In this case: *best_fit_exercise.out*

Part of the output file is shown below

Vector of solutions x

-1.359437853e-03

1.317862219e+00

Vector of residuals v

-0.002138

0.000162

0.001837

0.004354

-0.004216

The slope of the line is $b = \tan\theta = -1.359437853e-03$ and $\theta = -0^\circ 04' 40''$ (angles are considered +^{ve} anti-clockwise from the x-axis) and the bearing of the line of best fit will be $90^\circ 04' 40''$

The offset from the traverse to the line of best fit at B is 1.318 m (the value of c)

The offset from the traverse to the line of best fit at A is 1.046 m

Part of MATLAB function *least_squares.m*

Read the data file into memory

```
filepath = strcat('d:\temp\', '*.txt');
[infilename, inpathname] = uigetfile(filepath);
infilepath = strcat(inpathname, infilename);
rootName    = strtok(infilename, '.');
outfilepath = strcat(inpathname, outfilepath);

%-----
% 1. Load the data into an array whose name is the rootName
% 2. set fileTemp = rootName
% 3. Copy columns of data into individual arrays
%-----
load(infilepath);
fileTemp = eval(rootName);
```

Establish the matrices **B**, **f** and **W**

```
% get the number of rows (n) and the number of columns (m)
[n,m] = size(fileTemp);
% set the number of unknowns
u = m-2;
% copy the data into B, f and w
B = fileTemp(:,1:u);
f = fileTemp(:,u+1);
w = fileTemp(:,m);
% set the elements of the weight matrix W
W = zeros(n,n);
for k = 1:n
    W(k,k) = w(k);
end
```

Form the normal equations $\mathbf{N}\mathbf{x} = \mathbf{t}$

% form the normal equation coefficient matrix N

```
N = B'*W*B;
```

% form the vector of numeric terms t

```
t = B'*W*f;
```

Solve the normal equations $\mathbf{x} = \mathbf{N}^{-1}\mathbf{t}$

% solve the system $\mathbf{N}\mathbf{x} = \mathbf{t}$ for the unknown parameters x

```
Ninv = inv(N);
```

```
x = Ninv*t;
```

Compute the residuals $\mathbf{v} = \mathbf{B}\mathbf{x} - \mathbf{f}$

% compute residuals

```
v = f - (B*x);
```

Compute the variance factor σ_0^2 and the cofactor matrices $\mathbf{Q}_{xx}, \mathbf{Q}_{vv}, \mathbf{Q}_{ll}$

```
% compute the variance factor
```

```
vfact = (f'*W*f - x'*t)/(n-u);
```

```
% compute the cofactor matrix of the adjusted quantities
```

```
Qxx = Ninv;
```

```
% compute the cofactor matrix of the residuals
```

```
Qvv = inv(W)-B*Ninv*B';
```

```
% compute the cofactor matrix of the adjusted quantities
```

```
Qll = inv(W)-Qvv;
```

Open the data file and print out results

```
% open the output file print the data
```

```
fidout = fopen(outfilepath,'wt');
```


The diagram shows the path of a ship in a navigation channel at a constant speed and heading. Navigation equipment on board automatically measures distances to transponders at three navigation beacons *A*, *B* and *C*. The measured distances have an estimated standard deviation of 1 metre and each set of measurements occur at 60-second time intervals.

The coordinates of the navigation beacons are

<i>A</i> : 10000.000 E	<i>B</i> : 13880.000 E	<i>C</i> : 15550.000 E
10000.000 N	11250.000 N	7160.000 N

Transponder measurements at 60-second intervals

Measurement Epoch	Transponder measurements to navigation beacons		
	A	B	C
1	4249.7	7768.6	7721.1
2	3876.1	7321.4	7288.5
3	3518.4	6872.2	6857.6
4	3193.3	6426.0	6429.1
5	2903.6	5982.6	6009.7
6	2664.0	5543.2	5596.6
7	2490.9	5107.7	5191.5
8	2392.9	4678.9	4797.1
9	2383.2	4253.4	4417.8
10	2463.0	3841.7	4050.9

Measurement Epoch	Transponder measurements to navigation beacons		
	A	B	C
11	2623.2	3435.6	3709.9
12	2849.0	3054.2	3395.8
13	3126.7	2692.9	3119.4
14	3446.9	2366.6	2891.1
15	3793.4	2096.4	2724.4
16	4166.0	1900.6	2630.9
17	4552.2	1804.7	2610.2
18	4956.2	1824.8	2677.4
19	5366.4	1959.6	2819.7
20	5785.0	2182.8	3023.5

The exercise is to establish the matrix forms of the primary and secondary measurement models suitable for processing the measurements in a Kalman Filter.

Primary Measurement Model

$$\mathbf{v}_k + \mathbf{B}_k \hat{\mathbf{x}}_k = \mathbf{f}_k$$

The measurements can be written as

$$\mathbf{l}_k + \mathbf{v}_k = \hat{\mathbf{l}}_k \quad (\text{a})$$

where \mathbf{l}_k is the $m,1$ vector of measurements (transponder distances), \mathbf{v}_k is the $m,1$ vector of residuals and $\hat{\mathbf{l}}_k$ are estimates of the true (but unknown) value of the measurements. $m = 3$ is the number of measurements at each epoch.

\hat{l}_k are non-linear functions of the coordinates of the ship and the beacons and

coordinates E, N of the beacons A, B and C and the filtered state coordinates \hat{E}_k, \hat{N}_k of the ship at time t_k

$$\begin{aligned}\hat{l}_j &= \hat{l}(\hat{E}_k, \hat{N}_k, E_j, N_j) \\ &= \sqrt{(\hat{E}_k - E_j)^2 + (\hat{N}_k - N_j)^2} \quad \text{for } j = A, B, C\end{aligned}$$

Expanding this equation into a series using Taylor's theorem, and ignoring higher-order terms gives

$$\hat{l} = l' + \frac{\partial \hat{l}}{\partial \hat{E}_k} (\hat{E}_k - E'_k) + \frac{\partial \hat{l}}{\partial \hat{N}_k} (\hat{N}_k - N'_k)$$

where E'_k, N'_k are approximate coordinates of the ship at t_k , l' is an approximate distance computed using E'_k, N'_k and the coordinates of the beacon, and the partial derivatives are

$$\begin{aligned} \frac{\partial \hat{l}}{\partial \hat{E}_k} &= \frac{E'_k - E_j}{l'_j} = d_j \\ \frac{\partial \hat{l}}{\partial \hat{N}_k} &= \frac{N'_k - N_j}{l'_j} = c_j \quad \text{for } j = A, B, C \end{aligned}$$

For a single distance we may re-arrange (a) as

$$v - \hat{l} = -l$$

and substituting the Taylor series approximation for \hat{l} and re-arranging gives

$$v_j - d_j \hat{E}_k - c_j \hat{N}_k = l'_j - l_j + (-d_j E'_k - c_j N'_k) \text{ for } j = A, B, C$$

This primary measurement model can be expressed in terms of the filtered state vector $\hat{\mathbf{x}}_k$ at time t_k in the matrix form as

$$\begin{aligned}
 & \begin{bmatrix} v_A \\ v_B \\ v_C \end{bmatrix}_k + \begin{bmatrix} -d_A & -c_A & 0 & 0 \\ -d_B & -c_B & 0 & 0 \\ -d_C & -c_C & 0 & 0 \end{bmatrix}_k \begin{bmatrix} \hat{E} \\ \hat{N} \\ \dot{E} \\ \dot{N} \end{bmatrix}_k \\
 &= \begin{bmatrix} l'_A - l_A \\ l'_B - l_B \\ l'_C - l_C \end{bmatrix}_k + \begin{bmatrix} -d_A & -c_A & 0 & 0 \\ -d_B & -c_B & 0 & 0 \\ -d_C & -c_C & 0 & 0 \end{bmatrix}_k \begin{bmatrix} E' \\ N' \\ \dot{E}' \\ \dot{N}' \end{bmatrix}_k
 \end{aligned}$$

(b)

or

$$\mathbf{v}_k + \mathbf{B}_k \hat{\mathbf{x}}_k = \mathbf{l}'_k - \mathbf{l}_k + \mathbf{B}_k \mathbf{x}'_k = \mathbf{f}_k$$

Secondary (or dynamic) model

$$\mathbf{x}_k = \mathbf{T}\mathbf{x}_{k-1} + \mathbf{v}_m \quad (c)$$

The model corrections \mathbf{v}_m can be written as the product of two matrices as

$$\mathbf{v}_m = \mathbf{H}\mathbf{w} \quad (d)$$

where \mathbf{w} is the system driving noise and \mathbf{H} is a coefficient matrix. Applying Propagation of Variances to (d) gives the cofactor matrix of the system driving noise

$$\mathbf{Q}_m = \mathbf{H}\mathbf{Q}_w\mathbf{H}^T \quad (e)$$

To determine the elements of the transition matrix **T** consider the position of the ship as a simple function of time, say $y = y(t)$. Now, expand the function $y(t)$ into a series about the point $t = t_k$ using Taylor's theorem

$$y(t) = y(t_k) + (t - t_k)\dot{y}(t_k) + \frac{(t - t_k)^2}{2!}\ddot{y}(t_k) \\ + \frac{(t - t_k)^3}{3!}\ddot{\ddot{y}}(t_k) + \dots$$

where $\dot{y}(t_k)$, $\ddot{y}(t_k)$, $\ddot{\ddot{y}}(t_k)$, etc. are derivatives of y with respect to t evaluated at $t = t_k$

Letting $t = t_k + \delta t$ and then $t - t_k = \delta t$ then

$$y(t_k + \delta t) = y(t_k) + \dot{y}(t_k)\delta t + \frac{\ddot{y}(t_k)}{2!}(\delta t)^2 + \frac{\ddot{\ddot{y}}(t_k)}{3!}(\delta t)^3 + \dots$$

We now have a power series expression for the continuous function $y(t)$ at the point $t = t_k + \delta t$

Similarly, assuming $\dot{y}(t)$, $\ddot{y}(t)$, etc. to be continuous functions of t then

$$\dot{y}(t_k + \delta t) = \dot{y}(t_k) + \ddot{y}(t_k)\delta t + \frac{\ddot{\ddot{y}}(t_k)}{2!}(\delta t)^2 + \dots$$

$$\ddot{y}(t_k + \delta t) = \ddot{y}(t_k) + \ddot{\ddot{y}}(t_k)\delta t$$

Now considering two time epochs t_k and t_{k-1} separated by a time interval δt we may combine the equations into general matrix forms

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix}_k = \begin{bmatrix} 1 & \delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{2}(\delta t)^2 \\ \delta t \end{bmatrix} [\ddot{y}]_{k-1} \quad (f)$$

and

$$\begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}_k = \begin{bmatrix} 1 & \delta t & \frac{1}{2}(\delta t)^2 \\ 0 & 1 & \delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \ddot{y} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{6}(\delta t)^3 \\ \frac{1}{2}(\delta t)^2 \\ \delta t \end{bmatrix} [\ddot{y}]_{k-1}$$

For this exercise, the state vector of the ship in the channel contains position (E_k, N_k) and velocity (\dot{E}_k, \dot{N}_k) and we choose equation (f) as representative of the secondary (or dynamic) model [see (c) and (d)]

$$\mathbf{x}_k = \mathbf{T}\mathbf{x}_{k-1} + \mathbf{H}\mathbf{w}$$

$$\begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \end{bmatrix}_k = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E \\ N \\ \dot{E} \\ \dot{N} \end{bmatrix}_{k-1} + \begin{bmatrix} \frac{1}{2}(\Delta t)^2 & 0 \\ 0 & \frac{1}{2}(\Delta t)^2 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{bmatrix} \ddot{E} \\ \ddot{N} \end{bmatrix}_{k-1}$$

In this exercise it is assumed that the system driving noise \mathbf{w} contains small random accelerations caused by the sea and wind conditions, the steering of the ship, the engine speed variation, etc.

For this exercise assume that the estimates of the variances of the system driving noise are $s_{\ddot{E}}^2$ and $s_{\ddot{N}}^2$ and the cofactor matrix of the system driving noise is

$$\mathbf{Q}_w = \begin{bmatrix} s_{\ddot{E}}^2 & 0 \\ 0 & s_{\ddot{N}}^2 \end{bmatrix}$$